Section 5.1

Math 231

Hope College

Eigenvectors and Eigenvalues

- Let V be a vector space and $f : V \to V$ a linear transformation. A nonzero vector $\mathbf{x} \in V$ such that $f(\mathbf{x}) = \lambda\mathbf{x}$ for some scalar λ is called an **eigenvector** of f. The scalar λ is called the **eigenvalue** of f associated to the eigenvector $\mathbf{x}$.
- We will see several examples of eigenvalues and eigenvectors in class.

Eigenspaces

- Given an eigenvalue λ of f, we define

$$E_\lambda = \{\mathbf{x} \in V \mid f(\mathbf{x}) = \lambda\mathbf{x}\}.$$

  The set E_λ is called the **eigenspace** associated to the eigenvalue λ.
- **Theorem 5.7:** For every eigenvalue λ of f, the set E_λ is a subspace of V.

Linear Independence of Eigenvectors

- **Theorem 5.5:** Let $f : V \to V$ be a linear transformation on a vector space V.
  1. Let $S = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ be a set of eigenvectors of f with associated eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, respectively. If the numbers λ_j are all distinct, then S is a linearly independent set.
  2. If $\dim V = n$, then f has at most n distinct eigenvalues.
- A consequence of this theorem is that if $\dim V = n$ and f has n distinct eigenvalues, the set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ of eigenvectors associated to these eigenvalues will be a basis of V.

Finding Eigenvalues

- Given a square matrix A, we can find the eigenvalues of the matrix A, that is, the eigenvalues of the linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ for all $\vec{\mathbf{x}} \in \mathbb{R}^n$.
- We find the eigenvalues of A by solving $p_A(\lambda) = 0$, where $p_A(\lambda)$ is the **characteristic polynomial** of A:

$$p_A(\lambda) = \det(A - \lambda I_n).$$

- If V is a finite dimensional vector space and $f : V \to V$ is a linear transformation, then the eigenvalues of f can be found using the matrix $[f]_{\mathcal{B}}^{\mathcal{B}}$ for any basis $\mathcal{B}$ of V.